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Solvability of a one-dimensional quasilinear problem under nonresonance conditions on the potential

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Abstract

Problem of the type $-\Delta_p u = f(u) + h(x)$ in (a,b) with $u = 0$ on $\{a,b\}$ is solved under nonresonance conditions stated with respect to the first eigenvalue and the first curve in the Fučik spectrum of $(-\Delta_p, W_0^{1,p}(a,b))$, only on a primitive of f.

Keywords: One-dimensional p-Laplacian, Fučik spectrum, Nonresonance, Time-mapping, Degree theory. [1]

1 Introduction

This paper is mainly concerned with the following quasilinear two-point boundary value problem

$$(P) = \begin{cases} -(\varphi_p(u'))' = f(u) + h(x) \qquad \text{in} \quad (a,b) \\ u = 0 \qquad \text{on} \quad \{a,b\} \end{cases}$$

where $\varphi_p : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $\varphi_p(s) = |s|^{p-2}s$, with $p \in]1,+\infty]$, $f : \mathbb{R} \longrightarrow \mathbb{R}$, is a continuous function and $h \in L^1(a,b)$.

[1] *AMS subject classification: 34B15*

We denote by $\sum$ the set of couples of positive numbers (μ_+, μ_-) such that the homogeneous problem

$$(P_{\sum}) = \begin{cases} -(\varphi_p(u'))' = \mu_+\varphi_p(u^+) - \mu_-\varphi_p(u^-) \quad \text{in} \quad (a,b) \\ u = 0 \qquad \text{on} \quad \{a,b\} \end{cases}$$

has a nontrivial solution u. Here $u^+ = max(u,0)$, $u^- = u^+ - u$. The set $\sum$ is called the Fučik spectrum of the p-Laplacian operator $-\Delta_p$ on $W_0^{1,p}(a,b)$. Denote respectively by λ_1 and λ_2 the first and the second eigenvalue of $-\Delta_p$ on $W_0^{1,p}(a,b)$. It is well known that $\sum$ is composed of two trivial lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$, and of a sequence of hyperbolic-like curves (cf [2],[6]).The first curve C_1 passes through λ_2 and is the set

$$C_1 = \{(\mu_+, \mu_-) \in \mathbb{R}^2, 1/(\mu_+)^{1/p} + 1/(\mu_-)^{1/p} = \frac{b-a}{\pi_p}\}$$

where $\pi_p = 2(p-1)^{1/p} \int_0^1 \frac{ds}{(1-s^p)^{1/p}}$.

Let us denote by F the primitive of f defined by $F(s) = \int_a^s f(t)\,dt$. In some previous works (see for instance [1],[3],[4], [10]) many authors have proved the solvability of (P) when $h \in L^\infty(a,b)$ under various nonresonance assumptions on either the nonlinearity f, or on the primitive F, or on both f and F. As far as non-resonance conditions are considered at the right of λ_1, the Dolph-type condition:

$$\lambda_1 < \liminf_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} \leq \limsup_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} \leq \lambda_2 \tag{1}$$

is sufficient to yield solvability of (P) when $h \in L^\infty(a,b)$ (See [1]). It was observed in a recent work in [3], that weaker conditions with respect to the first curve in the Fučik spectrum such as

$$\lambda_1 < \limsup_{s\to\pm\infty} \frac{pF(s)}{|s|^p} \leq \limsup_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} \leq \mu_\pm \tag{2}$$

and

$$\liminf_{\substack{s\to+\infty \\ \text{or} \\ s\to-\infty}} \frac{pF(s)}{|s|^p} < \mu_+(\text{ or } \mu_-) \tag{3}$$

coupled with

$$\lim_{|s|\to+\infty} sgn(s)f(s) = +\infty$$

yield the same conclusion. Adapting an example given in [5], one can observe that assumption (3) cannot be relaxed to

$$\liminf_{\substack{s\to+\infty \\ \text{or} \\ s\to-\infty}} \frac{f(s)}{|s|^{p-2}s} < \mu_+(\text{ or } \mu_-)$$

Our purpose in the present paper is to weaken nonresonance conditions (2) and (3) at the light of a recent contribution in [11] for $p = 2$. Indeed, in [11] the solvability of (P) when $p = 2$ occurs under assumptions such as

$$\lim_{|s|\to+\infty} sgn(s)f(s) = +\infty \tag{4}$$

$$\liminf_{s\to\pm\infty} \frac{2F(s)}{s^2} > \mu_1 \tag{5}$$

and

$$\liminf_{s\to+\infty} \frac{2F(s)}{s^2} = \mu, \qquad \limsup_{s\to-\infty} \frac{2F(s)}{s^2} = \nu \tag{6}$$

where μ_1 is the first eigenvalue of $-\Delta$, on $H_0^1(a,b)$ and (μ,ν) is such that $\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}} > b - a$. It is worth noticing that the roles of s at infinity in (6) are interchangeable. Clearly, assumption such as (6)improves (3) in the particular case of $p = 2$ and the question naturally arises to know whether similar assumption can be extended to the p-Laplacian. The aim of this work is to investigate such a problem and as a result of this investigation we have the following.

Theorem 1.1. *Assume that*

(h_1) $\lim_{|s|\to+\infty} sgn(s)f(s) = +\infty$

(h_2) $\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} > \lambda_1$

(h_3) $\liminf_{s\to+\infty} \frac{pF(s)}{|s|^p} = \mu, \qquad \limsup_{s\to-\infty} \frac{pF(s)}{|s|^p} = \nu$

or $\liminf_{s\to-\infty} \frac{pF(s)}{|s|^p} = \mu, \qquad \limsup_{s\to+\infty} \frac{pF(s)}{|s|^p} = \nu$

with $1/\mu^{1/p} + 1/\nu^{1/p} > \frac{b-a}{\pi_p}$

Then problem (P) is solvable for any $h \in L^1(a,b)$.

As a consequence of our main result we have the following

Corollary 1.1. *Assume that*

$$\lim_{|s|\to+\infty} sgn(s)f(s) = +\infty$$
$$\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} > \lambda_1$$
and $\liminf_{s\to+\infty} \frac{pF(s)}{|s|^p} < \lambda_2, \qquad \limsup_{s\to-\infty} \frac{pF(s)}{|s|^p} \leq \lambda_2$

or $\liminf_{s\to+\infty} \frac{pF(s)}{|s|^p} \leq \lambda_2, \qquad \limsup_{s\to-\infty} \frac{pF(s)}{|s|^p} < \lambda_2$

Then problem (P) is solvable for any $h \in L^1(a,b)$.

Needless to mention that the limits at ∞ are interchangeable.
Thus, our result improves [3] in what concerns the conditions with respect to the first curve in the Fučik spectrum.
The proof of Theorem 1-1 is given in section 4. Basically, it uses time-mapping estimates to yield the needed a-priori bounds for a suitable parametrized problem related to (P) and combines topological degree argument to conclude . Our section 2 is devoted to the establishment of general properties for quasilinear differential equations useful for the proof of our main result. In section 3 we have given new estimate results for the time-mapping related to the p-Laplacian and accordingly improved some estimate results stated in [10]. Those estimates play a central role in the proof of Theorem 1-1.

2 General properties

Here we give general results for a large class of parametrized quasilinear problem of the form

$$(Q_\gamma) = \begin{cases} -(\varphi_p(u'))' = \hat{f}(x,u,\gamma) + \gamma h(x) \qquad \text{in} \quad (a,b), \gamma \in [0,1] \\ \\ u = 0 \qquad \text{on} \quad \{a,b\} \end{cases}$$

We assume that $h \in L^1(a,b)$ and $\hat{f} : [a,b] \times \mathbb{R} \times [0,1] \longrightarrow \mathbb{R}$ is a function satisfying the following:
(i) sign condition: $sgn(s)\hat{f}(x,s,\gamma) \geq -c$ for a positive constant c, for a.e. $x \in [a,b]$ and for $\gamma \in [0,1]$;
(ii) L^1-Carathéodory condition: $\hat{f}(x,.,\gamma)$ is continuous for a.e. $x \in [a,b]$, $\gamma \in [0,1]$, $\hat{f}(.,s,\gamma)$ is measurable for $s \in \mathbb{R}$ and $\gamma \in [0,1]$; moreover for each $R > 0$, there is $\Gamma_R \in L^1(a,b)$ such that $|\hat{f}(x,s,\gamma)| \leq \Gamma_R(x)$ for all $|s| \leq R$, a.e. $x \in [a,b]$, and for $\gamma \in [0,1]$. Solutions to (Q_γ) are intended in the sense that $u \in C^1[a,b]$, $\varphi_p(u')$ is absolutely continuous and u satisfies (Q_γ).

For any solution u of (Q_γ) we set here and henceforth the following

Definition 2.1. *We denote by x^* the first point of maximum of u and x_* the last point of minimum of u.*

Definition 2.2. *For any K such that $0 < K \leq \max u$, we denote*

$$\begin{aligned} \alpha_0 &= \max\{x \in [a, x^*), u(x) = 0\} \\ \beta_0 &= \min\{x \in (x^*, b], u(x) = 0\} \\ \alpha_K &= \min\{x \in [\alpha_0, x^*], u(x) = K\} \\ \beta_K &= \max\{x \in [x^*, \beta_0,], u(x) = K\} \end{aligned}$$

Definition 2.3. *For any K' such that $0 > K' \geq \min u$, we denote*

$$\begin{aligned} \alpha'_0 &= \max\{x \in [a, x_*), u(x) = 0\} \\ \beta'_0 &= \min\{x \in (x_*, b], u(x) = 0\} \\ \alpha'_{K'} &= \min\{x \in [\alpha'_0, x_*], u(x) = K'\} \\ \beta'_{K'} &= \max\{x \in [x_*, \beta'_0,], u(x) = K'\} \end{aligned}$$

Writing the first equation in (Q_γ) in the planar system

$$\varphi_p(u') = y(x) - \gamma \tilde{H}(x) \tag{7}$$

$$y'(x) = -\tilde{f}(x, u(x), \gamma) \tag{8}$$

with $\tilde{f}(x, s, \gamma) = \hat{f}(x, s, \gamma) + c$ and $\tilde{H}(x) = \int_a^x (h(t) - c)\, dt$, we derive the following.

Lemma 2.1. *A positive constant L exists such that any solution u of (Q_γ) satisfying $\max u > L$, fulfills the following conditions: there exist uniquely determined real numbers ρ and $\bar{\rho}$, with $\alpha_0 < \rho \leq x^* \leq \bar{\rho} < \beta_0$ such that*

(i)

$$y(x) > \|\tilde{H}\|_\infty \quad \text{on} \quad [\alpha_0, \rho)$$

$$|y(x)| \leq \|\tilde{H}\|_\infty \text{ on } [\rho, \bar{\rho}]$$

$$y(x) < -\|\tilde{H}\|_\infty \quad \text{on} \quad (\bar{\rho}, \beta_0]$$

(ii) u is strictly increasing on $[\alpha_0, \rho]$, strictly decreasing on $[\bar{\rho}, \beta_0]$ and

$$\max u - L \leq u(x) \leq \max u \quad \text{on} \quad [\rho, \bar{\rho}]$$

If furthermore

$$\lim_{s \to +\infty} \hat{f}(x, s, \gamma) = +\infty$$

uniformly for ($\gamma \in [0, 1]$, and a.e. $x \in [a, b]$), then for any $K > 0$ such that $K \leq \max u$ we have

(iii)
$$\lim_{K\to+\infty}(\alpha_K-\alpha_{K-L})=\lim_{K\to+\infty}(\beta_{K-L}-\beta_K)=0$$

Remark 2.1. *A dual version of Lemma (2.1)involving $\alpha_0', \beta_0', \alpha_{K'}', \beta_{K'}'$ can be obtained in the case that u is a solution of the planar system with $\min u < -L$ and*
$$\lim_{s\to-\infty}\hat{f}(x,s,\gamma)=-\infty$$
uniformly (for $\gamma\in[0,1]$ and a.e. $x\in[a,b]$). In this case $\tilde{f}$ and $\tilde{H}$ in the planar system are written $\tilde{f}(x,s,\gamma)=\hat{f}(x,s,\gamma)-c$, $\tilde{H}(x)=\int_a^x(h(t)+c)\,dt$.

Proof of Lemma 2.1

The proof of Lemma 2.1 in the particular case $p=2$ is given in [11]. We give here the general case for any $p>1$. So, let us consider α_0,β_0,x^* as set in the definitions 2.1, 2.2. Since $y'(x)=-\tilde{f}(x,u(x),\gamma)$, from the sign condition on $\hat{f}$, we have that y is strictly decreasing on (α_0,β_0) and accordingly
$$y(\alpha_0)>y(x^*)>y(\beta_0)$$
Moreover $u'(x^*)=0$, and then (7) yields
$$|y(x^*)|=|\varphi_p(0)+\gamma\tilde{H}(x^*)|\le\|\tilde{H}\|_\infty$$
Let $[\rho,\bar{\rho}]\subset[\alpha_0,\beta_0]$ be the maximal interval containing x^* and such that
$$|y(x)|\le\|\tilde{H}\|_\infty \tag{9}$$
Clearly, for such an interval, part (i) of Lemma 2.1 holds.
Since φ_p is a bijection on $\mathbb{R}$, one can write (7) on the form
$$u'(x)=\varphi_p^{-1}(y(x)-\gamma\tilde{H}(x))$$
And then using the monotoniciity of φ_p and (9), we have for $s\in[\rho,\bar{\rho}]$
$$|u'(x)|=|\varphi_p^{-1}(y(x)-\gamma\tilde{H}(x))|\le\varphi_p^{-1}(2\|\tilde{H}\|_\infty)$$
Accordingly, for x' and x'' in $[\rho,\bar{\rho}]$, we get
$$|u(x'')-u(x')|\le|\int_{x'}^{x''}u'(x)\,dx|\le(b-a)\varphi_p^{-1}(2\|\tilde{H}\|_\infty).$$
Consequently we get
$$\|u\|_\infty-(b-a)\varphi_p^{-1}(2\|\tilde{H}\|_\infty)\le u(x'')\le\|u\|_\infty\quad\text{for all}\quad x''\in[\rho,\bar{\rho}].$$

So by setting $L = (b-a)\varphi_p^{-1}(2\|\tilde{H}\|_\infty)$, we have part (ii) of the lemma.
To deal with part (iii) of the lemma, we note that since $\lim\limits_{s\to+\infty} \hat{f}(x,s,\gamma) = +\infty$ uniformly for $\gamma \in [0,1]$, and a.e. $x \in [a,b]$), for any $k > 0$, one can choose $v_k > 0$ large enough such that

$$\hat{f}(x,s,\gamma) \geq k \text{ for all } s \geq v_k, \gamma \in [0,1], \text{ and } a.e. \quad x \in [a,b].$$

Choose K with $K \geq v_k + L$ where $L = (b-a)\varphi_p^{-1}(2\|\tilde{H}\|_\infty)$.
Let's consider any solution of (Q_γ) such that $\max u \geq K$.
Since $K - L \leq u(x) \leq K$ when $x \in [\alpha_{K-L}, \alpha_K]$, we have

$$\tilde{f}(x,u(x),\gamma) \geq k, \quad \text{for all } x \in [\alpha_{K-L}, \alpha_K], \quad \text{and } \gamma \in [0,1] \tag{10}$$

and then

$$y(x) = y(\alpha_K) + \int_{\alpha_K}^{x} y'(t)\,dt$$

$$= \int_x^{\alpha_K} \tilde{f}(t,u(t),\gamma)\,dt \qquad \text{on} \quad [\alpha_{K-L}, \alpha_K].$$

Since $\max u \geq K$, we have $\alpha_K \in [\alpha_0, x^*]$ and then by using part (i) of the lemma and condition (10) we have

$$y(x) \geq -\|\tilde{H}\|_\infty + k(\alpha_K - x).$$

And then

$$u'(x) = \varphi_p^{-1}(y(x) - \gamma\tilde{H}(x)) \geq \varphi_p^{-1}(-2\|\tilde{H}\|_\infty + k(\alpha_K - x)) \quad \text{on } [\alpha_{K-L}, \alpha_K].$$

Next, we derive from the integration of u' on $[\alpha_{K-L}, \alpha_K]$ the following inequality

$$L = u(\alpha_K) - u(\alpha_{K-L}) \geq \int_{\alpha_{K-L}}^{\alpha_K} \varphi_p^{-1}(-2\|\tilde{H}\|_\infty + k(\alpha_K - x))\,dx \tag{11}$$

To go further with the integral in the left hand-side of (11), let us set

$$\Psi_p^*(s) = \int_0^s \varphi_p^{-1}(\xi)\,d\xi.$$

A simple computation shows that
$\varphi_p^{-1}(s) = s|s|^{\frac{2-p}{p-1}}$ for all $s \in \mathbb{R}$ and $p \in]1,+\infty[$ and then $\Psi_p^*(s) = \frac{p-1}{p}|s|^{\frac{p}{p-1}}$ for all $s \in \mathbb{R}$ and $p \in]1,+\infty[$. Clearly Ψ_p^* is an even strictly increasing function on $\mathbb{R}_+$. Let us denote by $\Psi_p^{*-1}(s)$ its positive inverse function on $\mathbb{R}_+$.

$$\Psi_p^{*-1}(s) = (\frac{p}{p-1})^{\frac{p-1}{p}} s^{\frac{p-1}{p}}, s \in \mathbb{R}_+.$$

On the other hand, the function

$$x \mapsto \Psi_p^*(-2\|\tilde{H}\|_\infty + k(\alpha_K - x))$$

is differentiable with respect to x and so

$$\frac{d}{dx}[\Psi_p^*(-2\|\tilde{H}\|_\infty + k(\alpha_K - x))] = -k\varphi_p^{-1}(-2\|\tilde{H}\|_\infty + k(\alpha_K - x)) \qquad (12)$$

Combining (11)and (12), we get

$$L \geq -\frac{1}{k}[\Psi_p^*(-2\|\tilde{H}\|_\infty + k(\alpha_K - x))]_{\alpha_{K-L}}^{\alpha_K}$$

and then

$$kL + \Psi_p^*(2\|\tilde{H}\|_\infty) \geq \Psi_p^*(-2\|\tilde{H}\|_\infty + k(\alpha_K - \alpha_{K-L})).$$

Now, using the positive inverse of Ψ_p^*, one has

$$\alpha_K - \alpha_{K-L} \leq \frac{\Psi_p^{*-1}(kL + \Psi_p^*(2\|\tilde{H}\|_\infty) + 2\|\tilde{H}\|_\infty}{k} \qquad (13)$$

One can easily see that the right hand-side of inequality (13) is equivalent to $k^{-1/p}$ at infinity. So when K tends to $+\infty$, k tends to $+\infty$, and then

$$\lim_{K \to +\infty} (\alpha_K - \alpha_{K-L}) = 0.$$

A similar argument on $[\beta_K, \beta_{K-L}]$ leads to

$$\lim_{K \to +\infty} (\beta_{K-L} - \beta_K) = 0.$$

So Lemma (2.1) is proved. ■

Remark 2.2. *When $K = \max u$, then $\alpha_K = \beta_K = x^*$ and then from (iii) of Lemma(2.1) we have* $\lim\limits_{u^* \to +\infty} (\beta_{\max u - L} - \alpha_{\max u - L}) = 0.$

Lemma 2.2. *Let u be a changing sign solution of (Q_γ) for $\gamma \in [0, 1]$ and let A, be a positive real number such that $\max u < A$ or $\min u > -A$ uniformly with respect to $\gamma \in [0, 1]$. Then a constant M (depending only on A) exists such that $\|u\|_\infty < M$.*

Proof Let us consider only the case $\max u < A$, the second case $\min u > -A$ of course can be proved similarly. Thus, suppose on the contrary that there exist a sequence $(\gamma_n) \in [0,1]$ denote (γ) for sake of simplicity of notation, and corresponding solutions u_n of (Q_γ), with $\max u_n < A$ and $\min u_n$ tending to $-\infty$. Then, from the sign condition and the L^1-Carathéodory condition on $\hat{f}$ we have

$$\hat{f}(x,u_n,\gamma) \leq c\chi_{\{u_n<0\}} + \Gamma_A \cdot \chi_{\{0\leq u_n<A\}} = \Gamma(x)$$

where for any set E, χ_E denote its characteristic function. Choose two points x_n^* and x_{*n} such that $u_n(x_n^*) = \max u_n$ and $u_n(x_{*n}) = \min u_n$. We can suppose without loss of generality that $x_n^* > x_{*n}$. Set $\tilde{u}_n = u_n - \bar{u}_n$ with $(x_n^* - x_{*n})\bar{u}_n = \int_{x_{*n}}^{x_n^*} u_n(x)\,dx$. Then after the multiplication of the first equation in (Q_γ) with $\tilde{u}_n$ and its integration over $[x_{*n}, x_n^*]$, we have

$$\int_{x_{*n}}^{x_n^*} |\tilde{u}_n'(x)|^p\,dx = \int_{x_{*n}}^{x_n^*} [\hat{f}(x,u_n,\gamma) - \Gamma(x)]\tilde{u}_n(x)\,dx + \int_{x_{*n}}^{x_n^*} [\gamma h(x) + (x)]\tilde{u}_n(x)\,dx$$

$$\leq ||\tilde{u}_n||_\infty \Big\{ \int_{x_{*n}}^{x_n^*} [-\hat{f}(x,u_n,\gamma_n) + \Gamma(x)]\,dx + ||h||_1 + \|\Gamma\|_1 \Big\}.$$

But

$$\int_{x_{*n}}^{x_n^*} [-\hat{f}(x,u_n,\gamma_n) + \gamma h(x)]\,dx = 0$$

and then we have

$$\int_{x_{*n}}^{x_n^*} |\tilde{u}_n'(x)|^p\,dx \leq 2[||h||_1 + ||\Gamma||_1]||\tilde{u}_n||_\infty \tag{14}$$

From the Hölder inequality we have

$$\left(\int_{x_{*n}}^{x_n^*} |\tilde{u}_n'(x)|\,dx\right)^p \leq (b-a)^{p-1} \int_{x_{*n}}^{x_n^*} |\tilde{u}_n'(x)|^p\,dx.$$

Combining the above inequality and (14), we get

$$(\max \tilde{u}_n - \min \tilde{u}_n) \leq 2(b-a)^{p-1}[||h||_1 + ||\Gamma||_1]||\tilde{u}_n||_\infty \tag{15}$$

Since $||\tilde{u}_n||_\infty^p \leq (\max \tilde{u}_n - \min \tilde{u}_n)$, inequality (15) yields

$$||u_n||_\infty \leq 2^{\frac{1}{p-1}}(b-a)[||h||_1 + ||\Gamma||_1]^{\frac{1}{p-1}}.$$

So the sequence $(\tilde{u}_n)$ is bounded and hence (u_n) is bounded. This is a contradiction to the fact that $\min u_n$ tends to $-\infty$. ■

3 Time-mapping and auxiliary functions

3.1 Time-mapping estimates

Let's consider the initial value problem

$$(I) = \begin{cases} -(\varphi_p(u'))' = g(u) \qquad \text{on} \quad \mathbb{R} \\ \\ u(0) = s, \qquad u'(0) = 0 \end{cases}$$

Where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $sgn(s)g(s) > -c$ for $c > 0$ and $G(s) \to +\infty$.
The function τ_g defined by

$$\tau_g(s) = 2c_p sgn(s) \int_0^s \frac{d\xi}{[G(s) - G(\xi)]^{1/p}} \qquad \text{for } s \text{ in } \mathbb{R}$$

with $G(s) = \int_0^s g(\xi)\, d\xi$, $c_p = \frac{1}{p^{*1/p}}$ and $p^* = \frac{p}{p-1}$ is the time-mapping associated to (I).
Under the assumptions $sgn(s)g(s) > -c$ for $c > 0$ and $G(s) \to +\infty$ when $|s| \to +\infty$, $\tau_g(s)$ is well defined for $|s|$ large enough. By adapting arguments developed in [12] for the case $p = 2$, one can easily derive that for s large enough (I) admits a periodic solution $u_s$with $||u_s||_\infty = s$ and $\tau_g(s)$ is the value of the half period. Time-mapping enables to provide a-priori estimates for solutions of boundary value problems (cf [7], [10], [11], [12]). Here, we give new results on the time-mapping estimates extending and even improving some results in [7], [10], [11], [12].

Lemma 3.1. *Assume that there exist positive real numbers $k^\pm$ and $k_\pm$ such that*

$$\limsup_{s\to\pm\infty} pG(s)/|s|^p = k^\pm \quad (\textit{resp.} \liminf_{s\to\pm\infty} pG(s)/|s|^p = k_\pm)$$

then

$$\liminf_{s\to\pm\infty} \tau_g(s) \geq \pi_p/(k^\pm)^{1/p} \quad (\textit{resp.} \limsup_{s\to\pm\infty} \tau_g(s) \leq \pi_p/(k_\pm)^{1/p})$$

Proof.
One can notice that under the assumption $sgn(s)g(s) > -c$ for $c > 0$ and the fact that $k^\pm$ $k_\pm$ are greater than 0, $G(s) \to +\infty$ when $|s| \to +\infty$ so that $\tau_g(s)$ is well defined for s large enough. Let's limit the proof of the lemma to the cases $\limsup\limits_{s\to-\infty} pG(s)/|s|^p = k^-$, (resp. $\liminf\limits_{s\to-\infty} pG(s)/|s|^p = k_-$), the other cases being similar.

For $s<0$ and for any ξ such that $|s|^p > |\xi|^p$, we have

$$\limsup_{s\to-\infty} \frac{pG(s)}{|s|^p - |\xi|^p}$$

$$= \limsup_{s\to-\infty} pG(s)/|s|^p \times \lim_{s\to-\infty} \frac{|s|^p}{|s|^p - |\xi|^p} = k^-$$

and then

$$\limsup_{s\to-\infty} \left[\frac{pG(s)}{|s|^p - |\xi|^p} - \frac{pG(\xi)}{|s|^p - |\xi|^p}\right] = k^-.$$

So for $\epsilon > 0$, there is a real number $s_0 < 0$ such that for $s < s_0$ we have

$$G(s) - G(\xi) \leq 1/p(k^- + \epsilon)(|s|^p - |\xi|^p).$$

Recalling the expression of $\tau_g(s)$ and taking into account inequality above, we get

$$\tau_g(s) \geq 2c_p \int_s^{s_0} \frac{d\xi}{[G(s) - G(\xi)]^{1/p}}$$

$$\geq \frac{2c_p p^{1/p}}{(k^- + \epsilon)^{1/p}} \int_s^{s_0} \frac{d\xi}{[|s|^p - |\xi|^p]^{1/p}}.$$

Setting $z = \xi/s$, one has

$$\tau_g(s) \geq \frac{2(p-1)^{1/p}}{(k^- + \epsilon)^{1/p}} \int_0^1 \frac{dz}{[1 - z^p]^{1/p}} = \frac{\pi_p}{(k^- + \epsilon)^{1/p}} \quad \text{for all } s < s_0 < 0.$$

Thus

$$\liminf_{s\to-\infty} \tau(s) \geq \frac{\pi_p}{(k^-)^{1/p}}.$$

For the case $\liminf\limits_{s\to-\infty} pG(s)/|s|^p = k_-$, we have
$\forall \epsilon > 0$, there is a real number $s_0 < 0$ such that for $s < s_0$

$$G(s) - G(\xi) \geq 1/p(k_- - \epsilon)(|s|^p - |\xi|^p), \quad \text{for all } s < \xi < 0.$$

So, for ϵ sufficiently small such that $k_- - \epsilon > 0$, we have

$$\tau_g(s) \leq \frac{2c_p p^{1/p}}{(k_- - \epsilon)^{1/p}} \int_s^{s_0} \frac{d\xi}{[|s|^p - |\xi|^p]^{1/p}}.$$

And by a simple computation as previously done we get

$$\limsup_{s\to-\infty} \tau_g(s) \leq \frac{\pi_p}{(k_-)^{1/p}}. \quad \blacksquare$$

3.2 Auxiliary functions related to the time-mapping

Let us consider the following parametrized problem

$$(P_\gamma) = \begin{cases} -(\varphi_p(u'))' = g(u,\gamma) + \gamma h(x) & \text{in} \quad (a,b), \gamma \in [0,1] \\ \\ u = 0 \qquad \text{on} \quad \{a,b\} \end{cases}$$

where $h \in L^1(a,b)$ and $g(.,\gamma) : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function for any $\gamma \in [0,1]$. (P_γ) is a particular type of problem (Q_γ) where the nonlinearity $\hat{f}$ does not depend on x. We assume that g satisfies the following sign condition: $\lim\limits_{|s|\to+\infty} sgn(s)g(s,\gamma) = +\infty$ uniformly with respect to $\gamma \in [0,1]$. From this sign condition, one can find a positive constant c such that $sgn(s)g(s,\gamma) > c$ for all $\gamma \in [0,1]$. Let us set

$$\tilde{H}(x) = \int_a^x (h(t) - c)\, dt \qquad \tilde{G}_\gamma(s) = \int_a^x \tilde{g}(\xi,\gamma)\, d\xi,$$

for each $\gamma \in [0,1]$, and where $\tilde{g}(s,\gamma) = g(s,\gamma) + c$ for $s \geq 0$ and $\tilde{g}(s,\gamma) = g(s,\gamma) - c$ for $s \leq 0$ The planar system equivalent to the first equation in (P_γ) is written

$$\varphi_p(u') = y(x) - \gamma\tilde{H}(x) \tag{16}$$
$$y'(x) = -\tilde{g}(u(x),\gamma) \tag{17}$$

for $x \in (a,b)$ and $\gamma \in [0,1]$. It is clear that the planar system (16), (17) is a particular case of the planar system (7), (8) and hence Lemma (2.1) is valid for any solution of (16), (17) as well.
For any solution u of (16), (17), let us consider the function T_ϵ where $\epsilon = \pm 1$ and defined by

$$T_\epsilon(x) = \frac{p-1}{p}|y(x) + \epsilon||\tilde{H}||_\infty|^{\frac{p}{p-1}} + \tilde{G}_\gamma(u(x)) \quad \text{on } [a,b]$$

with $\gamma \in [0,1]$ and $p > 1$. One can easily see that

$$T'_\epsilon(x) = y'(x)[|y(x) + \epsilon||\tilde{H}||_\infty|^{\frac{2-p}{p-1}}(y(x) + \epsilon||\tilde{H}||_\infty) - u'(x)].$$

Recalling part (i) of Lemma (2.1), we derive that:
for $\epsilon = -1$

$$T'_{-1}(x) = \begin{cases} y'(x)[(y(x) - ||\tilde{H}||_\infty)^{\frac{1}{p-1}} - u'(x)] & \text{on } [\alpha_0,\rho] \\ -y'(x)[(-y(x) + ||\tilde{H}||_\infty)^{\frac{1}{p-1}} + u'(x)] & \text{on } [\rho,\beta_0] \end{cases}$$

for $\epsilon = 1$

$$T_1'(x) = \begin{cases} y'(x)[(y(x) + ||\tilde{H}||_\infty)^{\frac{1}{p-1}} - u'(x)] & \text{on } [\alpha_0, \bar{\rho}] \\ -y'(x)[(-y(x) - ||\tilde{H}||_\infty)^{\frac{1}{p-1}} + u'(x)] & \text{on } [\bar{\rho}, \beta_0] \end{cases}$$

So recalling again part (i) of Lemma (2.1), one can easily check that

$$T_{-1}'(x) \geq 0 \quad \text{on } [\alpha_0, \rho] \qquad T_1'(x) \leq 0 \quad \text{on } [\rho, \beta_0] \tag{18}$$

Accordingly we have

$$\begin{aligned} T_{-1}(x) \leq T_{-1}(\rho) = \tilde{G}_\gamma(u(\rho)) \leq \tilde{G}_\gamma(\max u) \quad & \text{on } [\alpha_0, \rho] \\ T_1(x) \leq T_1(\bar{\rho}) = \tilde{G}_\gamma(u(\bar{\rho})) \leq \tilde{G}_\gamma(\max u) \quad & \text{on } [\bar{\rho}, \beta_0] \end{aligned} \tag{19}$$

Taking into account the expressions of $T_{-1}(x)$ and $T_1(x)$ and recalling again (i) of Lemma 2.1, we get

$$\begin{aligned} u'(x) \quad & \leq \varphi_p^{-1}[2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}] \quad \text{on } [\alpha_0, \rho] \\ -u'(x) \quad & \leq \varphi_p^{-1}[2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}] \quad \text{on } [\bar{\rho}, \beta_0] \end{aligned}$$

Next, by setting $\xi = u(x)$, we get

$$\begin{aligned} & (\beta_0 - \alpha_0) \geq (\beta_0 - \bar{\rho}) + (\rho - \alpha_0) \geq \\ c_p \int_0^{u(\bar{\rho})} & \frac{d\xi}{\varphi_p^{-1}[2{c_p}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}]} \\ +c_p \int_0^{u(\rho)} & \frac{d\xi}{\varphi_p^{-1}[2{c_p}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}]} \end{aligned} \tag{20}$$

From Lemma (2.1), $u(\rho)$ and $u(\bar{\rho})$ are greater than $\max u - L$ and then writing $\max u = s$ in (20), one has

$$(\beta_0 - \alpha_0) \geq 2c_p \int_0^{s-L} \frac{d\xi}{\varphi_p^{-1}[2{c_p}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]} \tag{21}$$

for $s > L$.
Considering the functions T_{-1} and T_1 respectively on $[\alpha_0', \rho']$ and $[\bar{\rho}', \beta_0']$ where $\alpha_0', \rho', \bar{\rho}', \beta_0'$ are the equivalents of $\alpha_0, \rho, \bar{\rho}, \beta_0$ in the dual version of Lemma (2.1),

and arguing as above with $\min u$ playing the role of $\max u$, we get

$$
\begin{aligned}
(\beta_0' - \alpha_0') \geq (\beta_0' - \bar{\rho}') + (\rho' - \alpha_0') \geq \\
c_p \int_{u(\bar{\rho}')}^{0} \frac{d\xi}{\varphi_p^{-1}[2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\min u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}]} \\
+c_p \int_{u(\rho')}^{0} \frac{d\xi}{\varphi_p^{-1}[2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\min u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}]}
\end{aligned}
$$

Next, writing $\min u = s$ and taking into account the fact that from the dual version of Lemma (2.1), $u(\rho')$ and $u(\bar{\rho}')$ are lower than $\min u + L$, one has

$$
(\beta_0' - \alpha_0') \geq 2c_p \int_{s+L}^{0} \frac{d\xi}{\varphi_p^{-1}[2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]} \tag{22}
$$

for $s < -L$.

In conclusion we have

$$
\begin{aligned}
(\beta_0 - \alpha_0) + (\beta_0' - \alpha_0') \geq \\
2c_p\sigma \sum_{\sigma\in\{-1,1\}} \int_{0}^{s-\sigma L} \frac{d\xi}{\varphi_p^{-1}[2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}
\end{aligned} \tag{23}
$$

for $|s| > L$.
Thus T_ϵ provides lower estimates for the length of the intervals $[\alpha_0, \beta_0]$ and $[\alpha_0', \beta_0']$.
Let us now deal with upper estimates provide by T_ϵ.
Going back to (18) and to the expressions of $T_1(x)$ and $T_{-1}(x)$ we derive that

$$
\begin{aligned}
T_1(x) \geq T_1(x^*) \geq \tilde{G}_\gamma(\max u) \quad &\text{on} \quad [\alpha_0, x^*] \\
T_{-1}(x) \geq T_{-1}(x^*) \geq \tilde{G}_\gamma(\max u) \quad &\text{on} \quad [x^*, \beta_0]
\end{aligned}
$$

and hence

$$
u'(x) \geq \varphi_p^{-1}[-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}] \quad \text{on } [\alpha_0, x^*] \tag{24}
$$

$$
-u'(x) \geq \varphi_p^{-1}[-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}}] \quad \text{on } [x^*, \beta_0] \tag{25}
$$

From part (ii) of Lemma (2.1), one has $[\alpha_0, \alpha_{\max u - L}] \subset [\alpha_0, x^*]$ and $[\beta_{\max u - L}, \beta_0] \subset [x^*, \beta_0]$. So let us consider inequalities in (24)and (25) respectively on $[\alpha_0, \alpha_{\max u - L}]$ and $[\beta_{\max u - L}, \beta_0]$ and let us assume that the following is satisfied (we will show farther in section 4 that such a condition is indeed satisfied under suitable condition):

$$-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}} > 0 \quad \text{on} \quad [\alpha_0, \alpha_{\max u - L}] \cup [\beta_{\max u - L}, \beta_0]. \tag{26}$$

Then, we derive after the change of variable $\xi = u(x)$ in (24) and (25), that

$$(\beta_0 - \beta_{\max u - L}) + (\alpha_{\max u - L} - \alpha_0) \leq 2c_p \int_0^{\max u - L} \frac{d\xi}{\varphi_p^{-1}[-2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}. \tag{27}$$

Arguing in a similar way, one can show that

$$(\beta'_0 - \beta'_{\min u + L}) + (\alpha'_{\min u + L} - \alpha'_0) \leq 2c_p \int_{\min u + L}^{0} \frac{d\xi}{\varphi_p^{-1}[-2c_p{}^{p-1}||\tilde{H}||_\infty + (\tilde{G}_\gamma(\min u) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}. \tag{28}$$

Now, let us set

$$T_\gamma(s) = 2c_p sgn(s) \int_0^{s - sgn(s)L} \frac{d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}$$

with $|s| > L$ and $K > 0$.

$$\tau_\gamma(s) = 2c_p sgn(s) \int_0^{s} \frac{d\xi}{[\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

with $|s| > 0$.

One can easily see that according to (21), (22)

$$(\beta_0 - \alpha_0) \geq T_\gamma(\max u) \quad \text{for} \quad \max u > L \tag{29}$$

$$(\beta'_0 - \alpha'_0) \geq T_\gamma(\min u) \quad \text{for} \quad \min u < -L \tag{30}$$

with $K = 2||\tilde{H}||_\infty$

On the other hand, the following lemma shows that $\tau_\gamma(s)$ is a good approximation of $T_\gamma(s)$ for s large enough.

Lemma 3.2. *Assume that* $\lim_{|s| \to +\infty} g(s, \gamma) = +\infty$ *uniformly with respect to* γ*. Assume that at least one of the functions* $T_\gamma(s)$ *and* $\tau_\gamma(s)$ *is uniformly bounded with respect to* γ*. Then*

$$\lim_{s \to \pm\infty} [T_\gamma(s) - \tau_\gamma(s)] = 0 \text{ uniformly with respect to } \gamma.$$

Proof Without loss of generality, we can suppose that it is $T_\gamma(s)$ which is bounded uniformly with respect to γ. Furthermore the proof will be given only for the case $s \to +\infty$, the case $s \to +\infty$ can be dealed similarly. So, let us consider

$$T_\gamma(s) \;=\; 2c_p \int_0^{s-L} \frac{d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}$$

for $s > L$

$$\tau_\gamma(s) \;=\; 2c_p \int_0^{s} \frac{d\xi}{[\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

for $s > 0$.

We observe that $K > 0$ implies

$$\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] \geq \varphi_p^{-1}[\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] = [\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}$$

for $0 < \xi < s$, and thus $\tau_\gamma(s) \geq T_\gamma(s)$ for $s > L$.
So, it suffices to show that for any $\epsilon > 0$, $\tau_\gamma(s) - T_\gamma(s) < \epsilon$ for s large enough.

Since $\lim\limits_{s\to+\infty} \tilde{g}(s,\gamma) = +\infty$ uniformly with respect to γ, for any $A > 0$ there exists a real number $d > 0$ such that

$$\tilde{g}(\xi,\gamma) \geq A \quad \text{for} \quad \xi \geq d \quad \text{and } \gamma \in [0,1] \tag{31}$$

and

$$\tilde{G}_\gamma(d) > \tilde{G}_\gamma(\xi) \quad \text{for} \quad 0 \leq \xi < d \quad \text{and } \gamma \in [0,1]. \tag{32}$$

Choose s such that $s > d + L$ with $L > 0$,

$$\tau_\gamma(s) - T_\gamma(s) \geq I = 2c_p \int_0^{s-L} \frac{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - [\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}\, d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

Let us split the integral I as follows

$$I = 2c_p \left[\int_0^d + \int_d^{s-L} \right] = 2c_p[I_1 + I_2].$$

Dealing with the first term of this decomposition, we get by using the monotonicity of φ_p^{-1}

$$I_1 \leq \int_0^d \frac{d\xi}{[\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}} \leq \int_0^d \frac{d\xi}{[\tilde{G}_\gamma(s) - \tilde{G}_\gamma(d)]^{\frac{1}{p}}}$$

Tending s to infinity, we notice that the right-hand side integral tends to zero. So for s large enough we have

$$2c_p I_1 < \frac{\epsilon}{2}.$$

To deal with the second term I_2,we write it as follows

$$I_2 = 2c_p \int_0^{s-L} \frac{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - \varphi_p^{-1}[(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]\, d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

In order to estimate I_2 the following inequalities will be useful.

Claim 1.

(i) A positive constant D exists such that for any real numbers a, b

$$|a-b|^p \leq D(|a|^{p-2}a - |b|^{p-2}b)(a-b) \quad \text{for} \quad p \geq 2$$

(ii) If a, b are non negative reals numbers then

$$(a+b)^p - b^p \leq pa(a+b)^{p-1} \quad \text{for} \quad p > 1$$

and

$$(a+b)^p - b^p \leq pab^{p-1} \quad \text{for} \quad 0 < p < 1$$

Proof.
For the case (i), on can refer to [9].
In order to prove (ii), let us consider the function

$$r(y) = (y+b)^p \quad \text{for} \quad 0 \leq y \leq a.$$

Obviously r is derivable and its derivative function $r'(y) = p(y+b)^{p-1}$ is increasing on [0,a]. So,

$$r(a) - r(0) = ar'(\xi) \leq pa(a+b)^{p-1} \quad \text{for} \quad 0 < \xi < a$$

Thus $(a+b)^p - b^p \leq pa(a+b)^{p-1}$.
The second inequality in (ii) follows similarly and thus claim (1) is proved. ■

Now, let's go ahead with the proof of the Lemma (3.2). Using (i) of claim (1) with $a = \varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]$ and $b = \varphi_p^{-1}[(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]$ we have

$$\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - \varphi_p^{-1}[(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] \leq (DK)^{\frac{1}{p-1}}$$

for $p \geq 2$.
For $1 < p < 2$, we have $\frac{1}{p-1} > 1$ and then we can apply the first inequality in (ii) of claim (1) with $\frac{1}{p-1}$ playing the role of p, $a = K$, $b = (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}$.
Thus, we have

$$[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]^{\frac{1}{p-1}} - [(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]^{\frac{1}{p-1}}$$
$$\leq \frac{1}{p-1} K[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]^{\frac{2-p}{p-1}},$$

that is

$$\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - \varphi_p^{-1}[(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]$$
$$\leq \frac{1}{p-1} K[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]^{\frac{2-p}{p-1}} \quad \text{for } 1 < p < 2.$$

In conclusion:
For $\quad p > 2$,

$$2c_p I_2 \leq 2c_p (DK)^{\frac{1}{p-1}} \int_d^{s-L} \frac{d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$
$$\leq \frac{(DK)^{\frac{1}{p-1}}}{[L \min \tilde{g}(\xi, \gamma)_{\xi \in [s-L, L]}]^{1/p}} \times T_\gamma(s) \leq \frac{(DK)^{\frac{1}{p-1}}}{[LA]}^{1/p} T_\gamma(s).$$

Since T_γ is uniformly bounded with respect to γ, by choosing A large enough we have $2c_p I_2 \leq \epsilon/2$.
For $\quad 1 < p < 2$

$$2c_p I_2 \leq \frac{2c_p K}{p-1} \int_d^{s-L} \frac{[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]^{\frac{2-p}{p-1}} \, d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$
$$\leq \frac{2c_p K}{p-1} \int_d^{s-L} \frac{[K/(\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}} + 1]^{\frac{2-p}{p-1}} \, d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi)]^{\frac{p-1}{p}}}$$
$$\leq \frac{2c_p K}{p-1} [\frac{K}{[L \min \tilde{g}(\xi, \gamma)_{\xi \in [s-L, L]}]^{\frac{p-1}{p}}} + 1]^{\frac{2-p}{p-1}} \times \frac{1}{[L \min \tilde{g}(\xi, \gamma)_{\xi \in [s-L, L]}]^{1/p}} \times$$
$$\int_d^{s-L} \frac{d\xi}{\varphi_p^{-1}[K + (\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}$$

$$\leq \frac{K}{p-1}\left[\frac{K}{(LA)^{\frac{p-1}{p}}}+1\right]^{\frac{2-p}{p-1}} \times \frac{1}{(LA)^{\frac{p-1}{p}}} \times T_\gamma(s).$$

Here again, for A large enough, we have $2c_pI_2 < \epsilon/2$ and finally, we get $2c_p(I_1+I_2) < \epsilon$ for s large enough, that is $\tau_\gamma(s)-T_\gamma(s) < \epsilon$ for s large enough. ■

An analogous of Lemma (3.2) holds when K is a negative real number. In order to state it let us start define

$$\tilde{T}_\gamma(s) = 2c_p sgn(s)\int_0^{s-sgn(s)L} \frac{d\xi}{\varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]} \quad \text{with } |s|>L \tag{33}$$

and K a negative real number.

$$\tilde{\tau}_\gamma(s) = 2c_p sgn(s)\int_0^{s-sgn(s)L} \frac{d\xi}{[\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi)]^{\frac{1}{p}}} \quad \text{with } |s|>L \tag{34}$$

It is worth noticing that according to (27) and (28)

$$(\beta_0-\beta_{\max u-L})+(\alpha_{\max u-L}-\alpha_0) \leq \tilde{T}_\gamma(\max u-L) \tag{35}$$
$$(\beta'_0-\beta'_{\min u+L})+(\alpha'_{\min u+L}-\alpha'_0) \leq \tilde{T}_\gamma(\min u+L) \tag{36}$$

with $K=-2||\tilde{H}||_\infty$.

Lemma 3.3. *Assume that* $\lim_{|s|\to+\infty} g(s,\gamma)=+\infty$ *uniformly with respect to* γ *and that at least one of the functions* $\tilde{T}_\gamma(s)$ *and* $\tilde{\tau}_\gamma(s)$ *is uniformly bounded with respect to* γ. *Moreover, suppose that the following condition is satisfied:*

$$(c_1) \quad K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}>0 \quad \textit{for } \xi\in[0,s-L],\ \textit{with } s>0$$

or

$$c_2) \quad K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}>0 \quad \textit{for } \xi\in[s+L,0],\ \textit{with } s<0.$$

Then respectively
$\lim_{s\to+\infty}[\tilde{T}_\gamma(s)-\tilde{\tau}_\gamma(s)]=0$ *or* $\lim_{s\to-\infty}[\tilde{T}_\gamma(s)-\tilde{\tau}_\gamma(s)]=0$ *uniformly with respect to* γ.

Proof. The proof is not too different of that of Lemma (3.2).We will sketch it below. Suppose that $\tilde{T}_\gamma(s)$ is uniformly bounded with respect to γ and let's give the proof when $s\to+\infty$ (the other cases being similar).

Since $\tilde{T}_\gamma(s) > \tilde{\tau}_\gamma(s)$ for $s > L$, we shall just have to prove that for any $\epsilon > 0$, $\tilde{T}_\gamma(s) - \tilde{\tau}_\gamma(s) < \epsilon$ for s sufficiently large.

$$\tilde{T}_\gamma(s)-\tilde{\tau}_\gamma(s) = 2c_p\int_0^{s-L}\frac{([\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi)]^{\frac{1}{p}} - \varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}])\,d\xi}{[\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi)]^{\frac{1}{p}}\varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]}$$

$$= 2c_p\left[\int_0^d + \int_d^{s-L}\right] = 2c_p[\tilde{I}_1+\tilde{I}_2]$$

with d as in (31)and (32). And then

$$\tilde{I}_1 \le \int_0^d \frac{d\xi}{\varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(d))^{\frac{p-1}{p}}]}.$$

So for s large enough, we have $2c_p\tilde{I}_1 < \epsilon/2$.
To estimate $\tilde{I}_2$ in the case $p \ge 2$, we proceed similarly as in the proof of Lemma (3.2) by using (i)of claim (1) to yield

$$\varphi_p^{-1}[(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - \varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] \le (-DK)^{\frac{1}{p-1}}$$

and next

$$2c_p\tilde{I}_2 \le \frac{(-DK)^{\frac{1}{p-1}}}{[LA]}^{1/p}\tilde{\tau}_\gamma(s).$$

Hence $2c_p\tilde{I}_2 \le \epsilon/2$ for s large enough and $p \ge 2$.
In the case $1 < p < 2$, following the same way as in Lemma (3.2), we apply the (ii) of claim(1) by writing the numerator of I_2 in the form

$$\varphi_p^{-1}[-K+K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}] - \varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}]$$

and by setting $a = -K$, $b = K + (\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}$; and then we obtain

$$2c_p\tilde{I}_2 \le -2c_p\frac{K}{p-1}\int_d^{s-L}\frac{(\varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}])^{2-p}\,d\xi}{\varphi_p^{-1}[K+(\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi))^{\frac{p-1}{p}}][\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

$$\le -2c_p\frac{K}{(p-1)(\varphi_p^{-1}[K+(L\min\tilde{g}(\xi,\gamma)_{\xi\in[s-L,L]})^{\frac{p-1}{p}}])^{p-1}}\int_d^{s-L}\frac{d\xi}{[\tilde{G}_\gamma(s)-\tilde{G}_\gamma(\xi)]^{\frac{1}{p}}}$$

So

$$2c_p\tilde{I}_2 \le -\frac{K}{(p-1)(\varphi_p^{-1}[K+(L\min\tilde{g}(\xi,\gamma)_{\xi\in[s-L,L]})^{\frac{p-1}{p}}])^{p-1}}\tilde{\tau}_\gamma(s)$$

$$\leq -\frac{K}{(p-1)(\varphi_p^{-1}[K+(LA)^{\frac{p-1}{p}}])^{p-1}}\tilde{\tau}_\gamma(s)$$

where A is as in (31). Since $\tilde{\tau}_\gamma(s)$ is uniformly bounded with respect to γ, we have $2c_p\tilde{I}_2 < \epsilon/2$ for s large enough and then $\tilde{T}_\gamma(s) - \tilde{\tau}_\gamma(s) < \epsilon$ for s sufficiently large. ■

4 Proof of the Theorem.

Let us consider the following parametrized problem

$$(S_\gamma) = \begin{cases} -(\varphi_p(u'))' = (1-\gamma)\theta\varphi_p(u) + \gamma[f(u)+h(x)] & \text{in} \quad (a,b) \\ u = 0 \quad \text{on} \quad \{a,b\} \end{cases}$$

where $\gamma \in [0,1]$ and θ is such that $\lambda_1 < \theta < \min(\mu_-, \mu_+)$.
Notice that the function defined by : $(s,\gamma) \mapsto (1-\gamma)\theta\varphi_p(s) + \gamma f(s)$ is a particular case of the function $\hat{f}$ and accordingly under the assumptions of Lemma (2.1)(respectively Lemma (2.2)), the conclusions of Lemma (2.1)(respectively Lemma (2.2)) are also valid for (S_γ) as well.
Under the assumptions (h_1), (h_3) of the theorem the following lemmas hold.

Lemma 4.1. *Under assumption (h_1) and the first part of assumption (h_3), that is*

$$\lim_{|s|\to+\infty} sgn(s)f(s) = +\infty$$

$$\liminf_{s\to+\infty} \frac{pF(s)}{|s|^p} = \mu, \qquad \limsup_{s\to-\infty} \frac{pF(s)}{|s|^p} = \nu$$

with

$$1/\mu^{1/p} + 1/\nu^{1/p} > \frac{b-a}{\pi_p}$$

(i) there exists a sequence $S_n \to +\infty$ such that if u solves (S_γ) for some $\gamma \in [0,1]$ and u changes sign, then $\max u \neq S_n$ for every n and every $\gamma \in [0,1]$.
(ii) When (h_1) and the second part of (h_3) hold, there exists a sequence $T_n \to -\infty$ such that if u solves (S_γ) for some $\gamma \in [0,1]$ and u changes sign, then $\min u \neq T_n$ for every n and every $\gamma \in [0,1]$.

Proof.
We prove only the first statement, the proof of the second one being similar. First let us denote by g the function :$(s,\gamma) \mapsto (1-\gamma)\theta\varphi_p(s) + \gamma f(s)$. According to

(h_1) $\lim_{|s|\to+\infty} sgn(s)g(s,\gamma) = +\infty$ uniformly with respect to γ and hence there is a positive constant c such that $sgn(s)g(s,\gamma) \geq -c$ for all $s \in \mathbb{R}$ and all $\gamma \in [0,1]$. Let $\tilde{G}_\gamma$ be the primitive such that

$$\tilde{G}_\gamma(s) = \begin{cases} \int_0^s (g(t)+c)\,dt & \text{for } s > 0 \\ \int_0^s (g(t)-c)\,dt & \text{for } s < 0. \end{cases}$$

For such a $\tilde{G}_\gamma$ we associate the function $\tilde{H}$ defined by

$$\tilde{H}(x) = \begin{cases} \int_a^x (h(t)-c)\,dt & x \in [a,b] & \text{when} & \tilde{G}_\gamma \text{ operates on } [0,+\infty[\\ \int_a^x (h(t)+c)\,dt & x \in [a,b] & \text{when} & \tilde{G}_\gamma \text{ operates on }]-\infty,0]. \end{cases}$$

From the first part of (h_3), we have

$$\liminf_{s\to+\infty} \frac{p\tilde{G}_\gamma(s)}{|s|^p} \leq \mu, \tag{37}$$

$$\limsup_{s\to-\infty} \frac{p\tilde{G}_\gamma(s)}{|s|^p} \leq \nu \tag{38}$$

Choose $\mu' > \mu$ such that the pair (μ', ν) still lies below C_1, that is

$$1/\mu'^{1/p} + 1/\nu^{1/p} > \frac{b-a}{\pi_p}.$$

For such a μ' we have

$$\limsup_{s\to+\infty}(\mu'\frac{s^p}{p} - \tilde{G}_\gamma(s)) = +\infty$$

Hence, there exists an increasing sequence $s_n \to +\infty$ so that for each n

$$p(\tilde{G}(s_n) - \tilde{G}(s)) \leq \mu'(s_n^p - s^p) \tag{39}$$

for all $s \in [0, s_n[$ with $s_n > L = (b-a)\varphi_p^{-1}(2||\tilde{H}||_\infty)$

Choose S_n as a tail sequence of the sequence s_n and suppose that with such a S_n, Lemma 4.1 is false. Then, one can find a subsequence of s_n still denoted by s_n, and solutions u_n of (S_γ) for $\gamma = \gamma_n \in [0,1]$ satisfying $\max u_n = s_n \to +\infty$, and hence according of Lemma (2.2), $\min u_n \to -\infty$. Let's show below that such a sequence solutions leads to a contradiction. So, let's consider the real numbers $(\alpha_0, \rho, \bar{\rho}, \beta_0)$ and $(\alpha_0', \rho', \bar{\rho'}, \beta_0')$ corresponding to the sequence solutions u_n as respectively in Lemma (2.1) and in its dual version. For the sake of simplicity we will keep the notations, $\alpha_0, \rho, \bar{\rho}, \beta_0$ and $\alpha_0', \rho', \bar{\rho'}, \beta_0'$, however those numbers depend on n . Recalling inequality (29), that is $(\beta_0 - \alpha_0) \geq T_\gamma(\max u_n)$ for

$\max u_n > L$ and using Lemma 3.2, we get
$\forall \epsilon > 0, \quad (\beta_0 - \alpha_0) \geq \tau_\gamma(S_n) - \epsilon$ for S_n large enough.
Next, combining (39) and inequality above, we get

$$(\beta_0 - \alpha_0) \geq \frac{2c_p p^{1/p}}{\mu'^{1/p}} \int_0^{S_n} \frac{d\xi}{[S_n^p - |\xi|^p]^{1/p}}$$

for S_n large enough.
And finally

$$\liminf_{n\to+\infty}(\beta_0 - \alpha_0) \geq \frac{\pi_p}{(\mu')^{1/p}} \tag{40}$$

To complete the proof, we need to estimate $(\beta_0' - \alpha_0')$. Recalling (30), we have $(\beta_0' - \alpha_0') \geq T_\gamma(\min u_n)$ for $|\min u_n|$ large enough and using again Lemma 3.2, we get

$$\forall \epsilon > 0, \quad (\beta_0' - \alpha_0') \geq \tau_\gamma(\min u_n) - \epsilon$$

for , $|\min u_n|$ large enough.
But, combining condition (38) and results in Lemma (3.1), we obtain

$$\liminf_{n\to+\infty}(\beta_0' - \alpha_0') \geq \liminf_{n\to+\infty} \tau_\gamma(\min u_n) \geq \frac{\pi_p}{(\nu)^{1/p}} \tag{41}$$

So, putting (40), (41) together yields

$$b - a \geq \liminf_{n\to+\infty}(\beta_0 - \alpha_0) + \liminf_{n\to+\infty}(\beta_0' - \alpha_0') > b - a.$$

This is a contradiction so Lemma 4.1 is proved. ■

The following lemma provides a-priori bounds for solutions of (S_γ) having a constant sign.

Lemma 4.2. *Assume that*

$$\lim_{|s|\to+\infty} sgn(s) f(s) = +\infty$$

and that condition (h_2) *holds, that is* $\liminf_{|s|\to+\infty} \frac{pF(s)}{|s|^p} > \lambda_1$.
Then, there are two constants $K > 0$, $K' < 0$ *such that there is no nonnegative solution or respectively no non-positive solution* u *of* (S_γ) *for some* $\gamma \in [0,1]$ *such that* $\max u \geq K$ *or respectively* $\min u \leq K'$.

Proof.
We give the proof only in the case of non negative solution, the case of non positive solution being similar.
Condition (h_2) gives

$$\liminf_{s\to+\infty} \frac{p\tilde{G}_\gamma(s)}{|s|^p} = k_+ > \lambda_1 \tag{42}$$

and accordingly from Lemma (3.1)

$$\limsup_{s\to+\infty} \tau_\gamma(s) \le \frac{\pi_p}{{k_+}^{1/p}} < b-a.$$

But since $\tilde{\tau}_\gamma(s) \le \tau_\gamma(s)$ for $s > L$ we also have

$$\limsup_{s\to+\infty} \tilde{\tau}_\gamma(s) \le \frac{\pi_p}{{k_+}^{1/p}} < b-a. \tag{43}$$

Suppose now that Lemma (4.2) is false, then we can find a sequence of non negative solutions (u_n) for some $\gamma \in [0,1]$ such that $\max u_n \to +\infty$. Let us show that such an assertion is absurd. So let us write the length $b-a = \beta_0 - \alpha_0$ as follows:

$$b-a = (\beta_0 - \beta_{\max u_n - L}) + (\alpha_{\max u - L} - \alpha_0) + (\beta_{\max u_n - L} - \alpha_{\max u - L}) \tag{44}$$

Since $\lim\limits_{s\to+\infty} sgn(s)g(s,\gamma) = +\infty$ uniformly with respect to γ, we have according to Lemma 2.1

$$\lim_{n\to+\infty} (\beta_{\max u_n - L} - \alpha_{\max u - L}) = 0 \tag{45}$$

In order to use (35) for estimating $(\beta_0 - \beta_{\max u_n - L}) + (\alpha_{\max u - L} - \alpha_0)$, we will prove here that inequality (26) previously admitted, that is

$$-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}} (\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}} > 0 \quad \text{on} \quad [\alpha_0, \alpha_{\max u - L}] \cup [\beta_{\max u - L}, \beta_0]$$

is effectively achieved under (42).
Indeed, under (42) we have the following: for $\epsilon > 0$, there is a positive real number $s_0 > 0$ such that for $s > s_0$

$$\tilde{G}_\gamma(s) - \tilde{G}_\gamma(\xi) \ge 1/p(k_+ - \epsilon)(s^p - \xi^p) \quad \text{for} \quad 0 < \xi < s.$$

So for n sufficiently large we have $\max u_n - L > s_0$ and then

$$\tilde{G}_\gamma(\max u_n) - \tilde{G}_\gamma(u_n(x)) \ge 1/p(k_+ - \epsilon)((\max u_n)^p - (u_n(x))^p)$$

for $\quad 0 < u_n(x) < \max u_n - L.$
So, for ϵ sufficiently small, we get

$$\tilde{G}_\gamma(\max u_n) - \tilde{G}_\gamma(u_n(x)) > 1/p\lambda_1((\max u_n)^p - (u_n(x))^p) \tag{46}$$

for $\quad 0 < u_n(x) < \max u_n - L.$
But $0 < u_n(x) < \max u_n - L$, implies

$$(\max u_n)^p - {u_n}^p(x) \geq L^p \text{ with } L = (b-a)\varphi_p^{-1}(2||\tilde{H}||_\infty) \tag{47}$$

Then, taking into account (46) and (47) we have

$$-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}} \geq$$

$$-2||\tilde{H}||_\infty + (\frac{p^*}{p})^{\frac{p-1}{p}}{\lambda_1}^{\frac{p-1}{p}}L^{p-1} =$$

$$-2||\tilde{H}||_\infty + (\frac{1}{p-1})^{\frac{p-1}{p}}(\frac{\pi_p}{b-a})^{p-1} \times (b-a)^{p-1}[\varphi_p^{-1}(2||\tilde{H}||_\infty)]^{p-1} =$$

$$2||\tilde{H}||_\infty\left(\frac{(\pi_p)^{p-1}}{(p-1)_{\frac{p-1}{p}}} - 1\right).$$

Noticing that

$$\pi_p = 2(p-1)^{1/p}\int_0^1 \frac{ds}{(1-s^p)^{1/p}} \geq 2(p-1)^{1/p}$$

for $p > 1$, we get

$$2||\tilde{H}||_\infty\left(\frac{(\pi_p)^{p-1}}{(p-1)_{\frac{p-1}{p}}} - 1\right) \geq 2||\tilde{H}||_\infty\left(2^{p-1} - 1\right) > 0$$

for $p > 1$.
In conclusion, it is proved that

$$-2||\tilde{H}||_\infty + (p^*)^{\frac{p-1}{p}}(\tilde{G}_\gamma(\max u) - \tilde{G}_\gamma(u(x)))^{\frac{p-1}{p}} > 0$$
$$\text{on} \quad [\alpha_0, \alpha_{\max u - L}] \cup [\beta_{\max u - L}, \beta_0]$$

which is of course inequality (26).
Hence, we can estimate $(\beta_0 - \beta_{\max u_n - L}) + (\alpha_{\max u - L} - \alpha_0)$ by using (35) and then we have $(\beta_0 - \beta_{\max u_n - L}) + (\alpha_{\max u_n - L} - \alpha_0) \leq \tilde{T}_\gamma(\max u_n)$ with K equals $-2||\tilde{H}||_\infty$ in $\tilde{T}_\gamma$. Since inequality (26) implies condition (c_1) of Lemma (4.1) with $K = -2||\tilde{H}||_\infty$, we have

$$(\beta_0 - \beta_{\max u_n - L}) + (\alpha_{\max u_n - L} - \alpha_0) \leq \tilde{\tau}_\gamma(\max u_n) + \epsilon$$

for all $\epsilon > 0$ and for $\max u_n$ large enough. Thus, combining (43),(44),(45) with the above inequality, one obtains

$$b - a \leq \limsup_{n \to +\infty} \tilde{\tau}_\gamma(\max u_n) + \epsilon < b - a + \epsilon$$

for all $\epsilon > 0$.
This is a contradiction and then Lemma (4.2) is proved. ■

We are now ready to introduce the functional analysis framework in which invariance of topological degree property will be used to conclude the proof of the theorem.
So, let us denote by

$$L : L^1(a, b) \to C^1[a, b]$$

the operator which sends $l \in L^1(a, b)$ on the unique solution of

$$(E_1) = \begin{cases} -(\varphi_p(u'))' = l \qquad \text{in} \quad (a, b) \\ \\ u(a) = u(b) = 0 \end{cases}$$

It is known that L is an odd and continuous operator (see [6]) and due to the compact embedding

$$J : C^1[a, b] \to C^0[a, b], \quad J \circ L : L^1(a, b) \to C^0[a, b]$$

is completely continuous. Moreover, for each $\gamma \in [0, 1]$, denote by
$K_\gamma : C^0[a, b] \to L^1(a, b)$ the operator defined by $K_\gamma(u) = g(., u, \gamma)$ with $K_\gamma(u)(x) = g(x, u, \gamma) = (1 - \gamma)\theta\varphi_p(u) + \gamma[f(u) + h(x)]$. Clearly K_γ is continuous and map bounded sets into bounded sets, hence for each $\gamma \in [0, 1]$, the operator

$$T_\gamma = J \circ L \circ K_\gamma : C^0[a, b] \to C^0[a, b]$$

is completely continuous and its fixed points are exactly the solutions of (S_γ). Moreover,for $\gamma = 0$, K_0 is an odd operator . Now let us built a suitable open bounded subset Ω of $C^0[a, b]$ on which the degree of T_γ, $\gamma \in [0, 1]$ is different of zero. The construction of Ω involves some constants provided by the different Lemmas (2.2), (4.1) (4.2). We consider first our theorem in the case that conditions $(h_1), (h_2)$ and the first part of (h_3) are satisfied. So, choose a constant K according to Lemma (4.2). Next, for n large enough, choose an element S_n denoted S of the sequence (S_n) such that $S > K$. For any possible changing sign solution u of (S_γ) for some $\gamma \in [0,1]$ such that $\max u < S$, Lemma (2.2) provides with positive real numbers such that $\min u > -M$. Take $-R = \min(-M, K')$ for a fixed M where $K' < 0$ is chosen according to Lemma (4.2).

$$\Omega = \left\{u \in C^0[a,b], -R < u(x) < S, \forall x \in [a,b]\right\}$$

The set Ω is such that $T_\gamma(u) \neq u$ for every $u \in \partial\Omega$ and $\gamma \in [0,1]$. Hence, by the homotopy invariance of the topological degree

$$deg(I - T_1, \Omega, 0) = deg(I - T_0, \Omega, 0)$$

where I is the identity operator in $C^0[a,b]$.

Claim 2. $deg(I - T_0, \Omega, 0) \neq 0$

Proof

By definition of $T_\gamma, \gamma \in [0,1]$, $\quad u - T_0 u = 0$ if only if u is a solution of

$$(E_2) = \begin{cases} -(\varphi_p(u'))' = \theta\varphi_p(u) & \text{in} \quad (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

But since $\lambda_1 < \theta < \min(\mu, \nu)$, $u = 0$ is the unique solution of (E_2). Moreover T_0 is odd, therefore by the Borsuk theorem

$$deg(I - T_0, B_r, 0) \neq 0$$

where B_r is the open ball of center O and raduis r in $C^0[a,b]$. Take r such that $B_r \subset \Omega$, then $0 \notin (I - T_0)^{-1}(\bar{\Omega} \setminus B_r)$ and from the excision property of the degree, we have $deg(I - T_0, \Omega, 0) = deg(I - T_0, B_r, 0) \neq 0$. ■

Consequently $deg(I - T_1, \Omega, 0) \neq 0$ and by the existence property of the topological degree T_1 has a fixed point in Ω which in turn is precisely a solution of (P). In the context that it is the conditions (h_1), (h_2) and the second part of (h_3) which are satisfied, we construct our Ω with parameters S and R as follow: we fix $-R = T_n$ with $T_n < K' < 0$ for n large enough, where T_n is as in Lemma (4.1). Next, we choose $S = \max(M, K)$ for a fixed M where M is as in Lemma (2.2). A similar argument of topological degree as above yields again the solvability of (P) in this latter case. This completes the proof of the theorem. ■

Remark 4.1. *It is worth noticing that the establishment of Lemma (2.1) does not involve the boundary conditions so that Lemma (2.1) can be usefully employed in dealing with other boundary conditions. The time-mapping estimates in Lemma (3.1) and the auxiliary functions estimates in Lemma (3.2) and Lemma (3.3) are many tools which can be combined with others in order to extend to the one-dimensional p- Laplacian* $(p > 1)$, *many others results previously obtained for the Laplacian.*

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